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THERMOELASTIC FRACTURE SOLUTIONS USING DISTRIBUTIONS OF SINGULAR INFLUENCE FUNCTIONS—II

NUMERICAL MODELLING OF THERMALLY SELF-DRIVEN CRACKS

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Abstract—Propagation of an array of cracks, due to cooling of a body and crack surfaces has many important applications; e.g. it could greatly increase the rate at which heat can be extracted from a Hot Dry Rock geothermal system. A procedure and numerical program are presented for determining the stresses associated with such a crack array, assuming it is propagating under steady state conditions. The technique employs superposition of the stress fields associated with constant-velocity heat sinks and dislocation distributions can be used to simulate resulting cracking, as described in Part I of the paper.

INTRODUCTION

A proposed method for extracting heat from the near-surface rock in the earth's crust is the "Hot Dry Rock" geothermal system being investigated by Los Alamos Scientific Laboratory[1, 2]. In the proposed system, a well is drilled into impermeable rock to a depth at which the temperature is sufficiently high to be useful. At that depth, a large hydraulic fracture is created essentially normal to minimum principal stress, through which cold water is circulated to extract heat from the rock.

Cracks are expected to develop on the surface of this fracture due to the tensile stresses created when the rock cools. If the thermal cracks can be cooled sufficiently by the fluid flowing through them, they may be able to sustain propagation without the need for conductive cooling from the original fracture surface. Heat removal by such cracks would be much more rapid than that due to purely conductive heat transfer around the main hydraulic fracture. The purpose of this article is to develop a method for determining the conditions under which such self-propagating cracks are possible.

An infinite array of parallel, identical planar cracks, aligned such that their leading edges are all contained in a plane perpendicular to the crack planes, and subjected to uniform loading normal to the crack planes (see Fig. 1) has stress intensity factors which remain essentially unchanged once the crack lengths exceed twice the crack spacing[3, 4]. Thus, an array of cracks, self-driven by the cooling effect of the fluid flowing through them, can be expected to behave as though they are essentially semi-infinite in length once their lengths exceed twice their spacing. Also, assuming that the pattern of cooling remains the same as the cracks grow, they would propagate at a uniform velocity once they had reached this length.

The ability of such an infinite array of semi-infinite cracks to propagate at a constant velocity can be judged from a knowledge of their stress intensity factors and crack opening displacements. This information can be obtained through a procedure such as that described in Part I of this paper, once the crack surface tractions are known. The method presented in this paper for determining these tractions uses the superposition of the stress fields due to steadily moving line sources or sinks of heat. The superposition procedure is described first for a single crack and then for an infinite array of cracks.

For purposes of this paper, the material in which the cracks are propagating is assumed to be homogeneous, isotropic, and linearly elastic with Young's modulus E, Poisson's ratio ν, linear coefficient of thermal expansion α, mass density ρ, coefficient of thermal conductivity k, heat capacity C, and thermal diffusivity $c = k/\rho C$. The material is initially stress-free and at a uniform temperature T_0. The cracks are assumed to propagate with a velocity v in the $+y$-direction or, equivalently, the material moves with a speed v in the $-y$-direction past stationary cracks. This latter configuration will be assumed here. The problem is one of plane strain. Spatial dimensions are nondimensionalized by $b = 2c/v$, i.e. $X = x/b$ and $Y = y/b$.

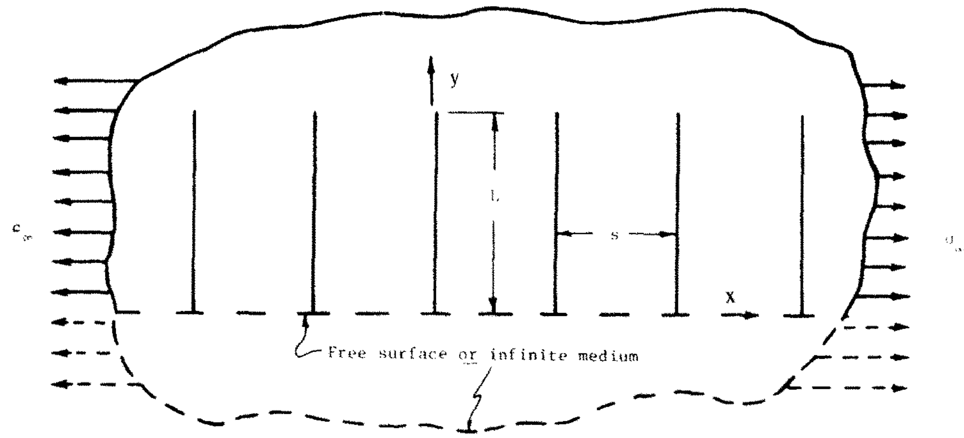

Fig. 1. An infinite array of parallel cracks loaded in mode I has stress intensity factors which remain constant for $L>2s$. Furthermore, this constant value is the same whether or not a free surface is present.

MOVING LINE SOURCE

Consider first a stationary line heat source located at coordinates (X_s, Y_s) emitting $\dot{Q}'$ units of heat per unit time and unit length. The material moves past the source with speed v in the $-y$-direction. Steady state conditions are assumed.

Carslaw and Jaeger[5] give the temperature at point (X, Y) due to this heat source as

$$\theta(X, Y) \equiv T(X, Y) - T_0 = \frac{\dot{Q}'}{2\pi k} U_\theta(X, Y; X_s, Y_s) \tag{1a}$$

where

$$U_\theta(X, Y; X_s, Y_s) = e^{-(Y-Y_s)} K_0(r), \tag{1b}$$

$r = [(X - X_s)^2 + (Y - Y_s)^2]^{1/2}$, and K_j represents the modified Bessel function of the second kind, of order j. U_θ will be referred to as the "temperature influence function" of the line heat source.

Cleary[6] gives the distribution of stresses for the analogous case of a moving line fluid source in a porous medium. His solution for the stress field due to such a source is based on the fact that

$$\sigma_{xx} + \sigma_{yy} + \beta P = 0 \tag{2}$$

where P is the fluid pressure and β is a material property. The corresponding equation in the plane strain thermoelastic context is the compatibility condition

$$\nabla^2\left(\sigma_{xx} + \sigma_{yy} + \frac{E\alpha}{(1-\nu)}\theta\right) \equiv \nabla^2\Phi = 0 \tag{3}$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right).$$

Replacing βP in the poroelastic stress solution by $E\alpha\theta/(1-\nu)$ for the thermoelastic case ensures that eqn (3) is satisfied since the solution requires that $\Phi = 0$. As can be verified from eqn (4) below, the equations of equilibrium are also satisfied. Since there is only one stress state for a linearly elastic body (undergoing infinitesimal strains) which satisfies the equilibrium and compatibility equations for a given set of boundary conditions, the solution obtained by the specialization $\Phi = 0$ is the correct one as long as the boundary conditions are satisfied. It will be seen that this requires the superposition of a uniform stress state (which, of course, also satisfies equilibrium and compatibility), for problems solved here.

The thermoelastic stresses associated with the moving line heat source, when Φ is required to vanish, are [6]

$$\begin{Bmatrix} \sigma_{xx}(X, Y) \\ \sigma_{yy}(X, Y) \\ \sigma_{xy}(X, Y) \end{Bmatrix} = \frac{-\dot{Q}'E\alpha}{4\pi k(1-\nu)} \begin{Bmatrix} e^{-(Y-Y_s)}\left[K_0(r) + \frac{(Y-Y_s)}{r}K_1(r)\right] + \frac{(Y-Y_s)}{r^2} \\ e^{-(Y-Y_s)}\left[K_0(r) - \frac{(Y-Y_s)}{r}K_1(r)\right] - \frac{(Y-Y_s)}{r^2} \\ -\left[e^{-(Y-Y_s)}K_1(r) - \frac{1}{r}\right]\frac{(X-X_s)}{r} \end{Bmatrix}. \tag{4}$$

Since the expression for σ_{xx} is used extensively in the following discussion, it is convenient to define a "stress influence function" for σ_{xx} as

$$U_\sigma(X, Y; X_s, Y_s) \equiv e^{-(Y-Y_s)}\left[K_0(r) + \frac{(Y-Y_s)}{r}K_1(r)\right] + \frac{(Y-Y_s)}{r^2}. \tag{5}$$

This influence function is plotted in Fig. 2. A feature of this stress field which could have important consequences is the fact that there is a region of tensile stress ahead of the moving source. Similarly, a moving heat sink would have a compressive region ahead of it. Thus if the thermal effect of a cooled, moving crack is represented by a distribution of heat sinks along the crack, there will be a compressive region ahead of the crack and possibly even compressive stresses acting to close the portion of the crack behind the crack tip.

SINGLE MOVING CRACK

Consider now the half-plane ($X = X_s$, $Y < 0$) acting as a heat source with the surrounding material moving in the $-y$-direction at a speed v, as shown in Fig. 3(a). At (X_s, Y_s), $Y_s < 0$, $\dot{Q}''(Y_s)$ units of heat per unit time and unit area are emitted.

The temperature and normal stress in the x-direction at point (X, Y) due to the heated half-plane are

$$\begin{Bmatrix} \theta(X, Y) \\ \sigma_{xx}(X, Y) \end{Bmatrix} = \frac{b}{2\pi k}\int_{-\infty}^{0} \mathrm{d}Y_s \dot{Q}''(Y_s) \begin{Bmatrix} U_\theta(X, Y; X_s, Y_s) \\ \frac{E\alpha}{2(1-\nu)}U_\sigma(X, Y; X_s, Y_s) \end{Bmatrix}. \tag{6}$$

The heated half-plane is now taken to represent a crack whose surface temperature has changed relative to T_0 by an amount $\theta(X_s, Y)$ due to the convective heat transfer associated with fluid flow through the crack. This situation can be treated as the superposition of two cases, one in which the material is cracked and one in which it remains uncracked.

In the uncracked case, the first of eqns (6) becomes an integral equation to be solved for the heat source distribution $\dot{Q}''(Y_s)$ along the proposed crack. This distribution is used in the second of eqns (6) to determine the normal tractions on the proposed crack surface. Shear tractions are absent due to symmetry.

For the cracked case, the crack surface tractions required are those which, when superimposed with those of the uncracked case, give the true crack tractions. When these tractions are known, the crack opening displacements and crack tip stress intensity factors can be computed using a procedure such as that explained in Part I, reasonably assuming that the velocities involved are small enough to make the problem elastically quasi-static. The remainder of this article will describe the computational method used to solve eqns (6) for the "crack" tractions in the uncracked case.

The heat source distribution $\dot{Q}''(Y_s)$ along the y-axis is approximated as the superposition of regions for which $\dot{Q}''$ is constant; the jth region is centered at $Y = Y_j$ and has amplitude Q_j and

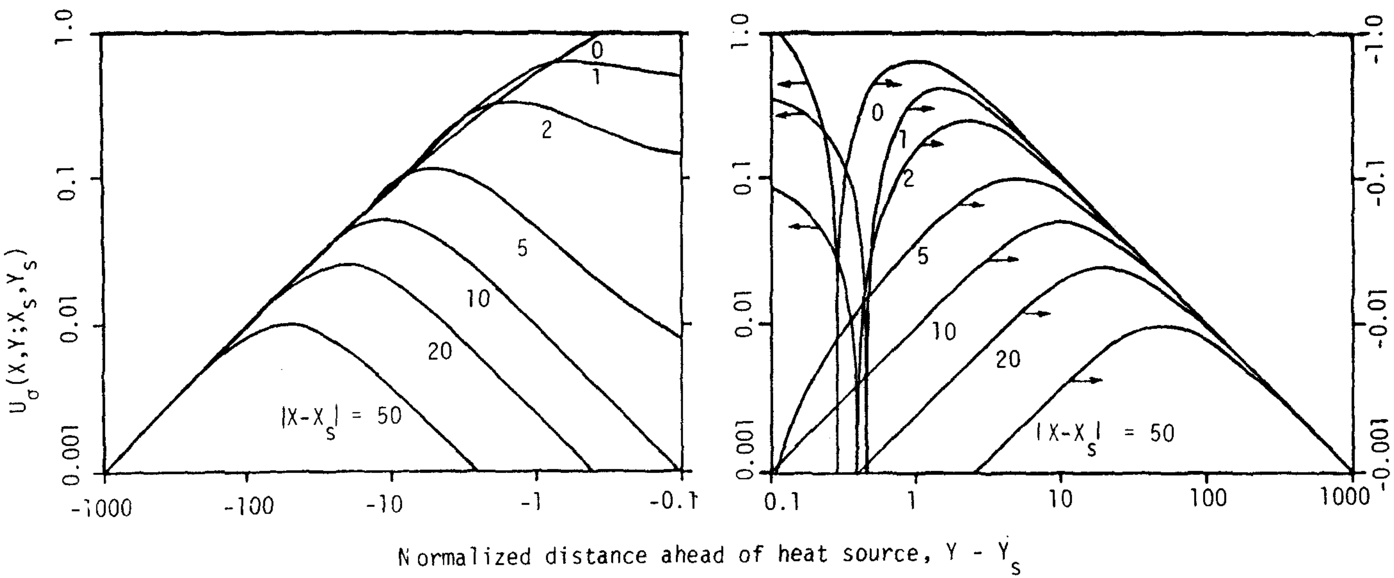

Fig. 2. Influence function for σ_{xx} due to constant-velocity heat source.

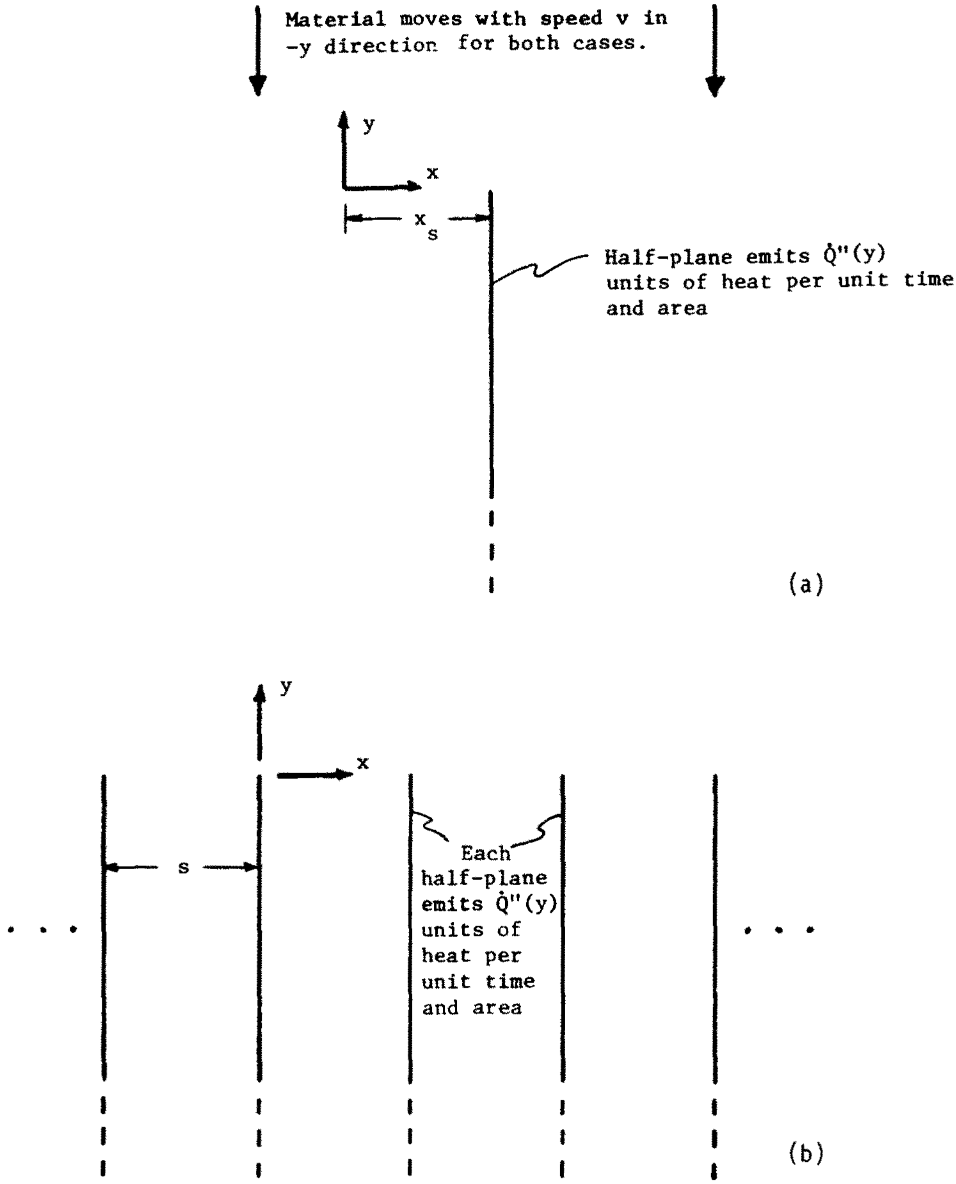

Fig. 3. Configurations for self driven crack problems. (a) Single semi-infinite crack. (b) Infinite array of parallel semi-infinite cracks.

width W_j. That is,

$$\dot{Q}''(Y_s) \simeq \frac{2\pi k}{b} \sum_{j=1}^{\infty} Q_j \delta_j, \quad (Y_s < 0) \tag{7a}$$

where

$$\delta_j = 1 \text{ if } Y_j - \frac{1}{2} W_j < Y_s \leq Y_j + \frac{1}{2} W_j, \ \delta_j = 0 \quad \text{otherwise} \tag{7b}$$

and

$$Y_1 = -\frac{1}{2} W_1, \quad Y_{j+1} = Y_j - \frac{1}{2}(W_j + W_{j+1}). \tag{7c}$$

Using eqn (7) in eqn (6) and changing the integration variable to $\eta = 2(Y_s - Y_j)/W_j$ gives

$$\begin{Bmatrix} \theta(X, Y) \\ \sigma_{xx}(X, Y) \end{Bmatrix} \simeq \sum_{j=1}^{\infty} Q_j \begin{Bmatrix} C_j(X, Y, X_s) \\ D_j(X, Y, X_s) \end{Bmatrix} \tag{8a}$$

where

$$\begin{Bmatrix} C_j(X, Y, X_s) \\ D_j(X, Y, X_s) \end{Bmatrix} = W_j \begin{Bmatrix} 1/2 \\ \dfrac{E\alpha}{4(1-\nu)} \end{Bmatrix} \int_{-1}^{1} d\eta \begin{Bmatrix} U_\theta(X, Y; X_s, Y_{sj}) \\ U_\sigma(X, Y; X_s, Y_{sj}) \end{Bmatrix} \tag{8b}$$

and $Y_{sj} = Y_j + \eta W_j/2$.

In actual computations the summations are truncated after a finite number of terms, M, giving a finite crack length. This crack length must be great enough to give stresses and crack opening displacements in the near-tip region of interest which do not change appreciably as the crack length is increased further.

The first of eqns (8a) is solved for the Q_j with $\theta(0, Y)$ being evaluated at the points $Y_i = Y_j$. The normal crack tractions $\sigma_{xx}(0, Y)$ are also calculated at these points to minimize computation time (since U_θ is required in computing U_σ).

Equations (8) then give two matrix equations

$$\{\theta(0, Y_i)\} \simeq [C^\circ_{ij}]\{Q_j\}, \quad C^\circ_{ij} = C_j(0, Y_i, 0) \tag{9a}$$

$$\{\sigma_{xx}(0, Y_i)\} \simeq [D^\circ_{ij}]\{Q_j\}, \quad D^\circ_{ij} = D_j(0, Y_i, 0). \tag{9b}$$

The first of these is to be solved for Q_j, $j = 1, 2, \ldots, M$ and the second then provides $\sigma_{xx}(0, Y_i)$, $i = 1, 2, \ldots, M$.

The integrals in eqn (8b) are evaluated numerically. However, since the influence functions both behave as $-\ln(r/2)$ as r approaches zero (i.e. as η approaches zero with $i = j$), accurate numerical integration of them is difficult when $i = j$. For these cases, the influence functions are expressed (using U to represent either U_θ or U_σ) as $U = U^* - \ln(r/2)$ where U^* has no singular behavior and can be accurately integrated numerically. The integrals of $\ln(r/2)$ are evaluated analytically.

For the influence functions given in eqns (1b, 5) it was found that satisfactory accuracy was obtained when all $W_j \leq 25$ and the number of collocation points $J = 4(i \neq j)$, $32(i = j, U^* = U_\theta^*)$, $64(i = j, U^* = U_\sigma^*)$. W_1 was 10^{-4} and successive W_j increased geometrically up to the maximum.

INFINITE ARRAY OF SELF-DRIVEN CRACKS

For the case of an infinite number of identical, parallel self-driven cracks, with crack n having its tip located at the coordinates $(X, Y) = (ns^*, 0)$, $n = 0, \pm 1, \pm 2, \ldots, s^* = s/b$ (see Fig. 3b), eqn (8) can be extended to get

$$\begin{Bmatrix} \theta(X, Y) \\ \sigma_{xx}(X, Y) \end{Bmatrix} \simeq \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} Q_j \begin{Bmatrix} C_j(X, Y, ns^*) \\ D_j(X, Y, ns^*) \end{Bmatrix}. \tag{10a, b}$$

The temperature influence function, for large n, has the behavior

$$U_\theta(0, Y_i; ns^*, Y_{sj}) \sim \exp\{(Y_{sj} - Y_i) - Z\}/Z^{1/2}$$

where $Z = [(Y_i - Y_{sj})^2 + n^2 s^{*2}]^{1/2}$. Thus for values of n such that $Z \gg 30 + (Y_{sj} - Y_i)$, U_θ is negligibly small.

The influence function for stress, however, has the behavior

$$U_\sigma(0, Y_i; ns^*, Y_{sj}) \sim (Y_{sj} - Y_i)(ns^*)^{-2} \tag{11a}$$

for large values of $|n|$ and therefore requires a much larger value of $|n|$ than U_θ does before contributions become negligible. Fortunately, not all these cracks need to be considered explicitly, because once the asymptotic behavior is achieved, the Euler–Maclaurin summation formula is applied, ([6], no. 3.6.28) for the contributions of the remaining cracks. Thus, if eqn (11a) is realized for a particular pair of values for i and j when $|n| > n^*_{ij}$, then

$$\int_{-1}^{1} \mathrm{d}\eta U_\sigma(0, Y_i; ns^*, Y_{sj}) \sim 2(Y_{sj} - Y_i)(ns^*)^{-2} \tag{11b}$$

for $|n| > n^*_{ij}$ and eqn (10b) becomes, for $X = 0$ and a given value of i,

$$\sigma_{xx}(0, Y_i) \simeq \sum_{j=1}^{\infty} Q_j \left\{ \sum_{n=-n_{ij}^*}^{n_{ij}^*} D_j(0, Y_i, ns^*) - \frac{(Y_{sj} - Y_i)}{(n^*_{ij}s^*)^2} \frac{\alpha E W_i}{(1-\nu)} \left[n_{ij}^* - \frac{1}{2} + \frac{1}{6n_{ij}^*} - \frac{1}{30n_{ij}^{*3}} + \frac{1}{42n_{ij}^{*5}} - \frac{1}{30n_{ij}^{*7}} \pm \cdots \right] \right\}.$$

Since $\theta(ns^*, Y)$ is the same for all n, eqns (10) need only be solved for $X = 0$. $\theta(0, Y_i)$ is specified and eqn (10a) is solved for the Q_j. Equation (10b) then gives $\sigma_{xx}(0, Y)$.

A listing of a Fortran computer program written to formulate and solve eqns (10) can be found in reference [10].

As stated previously, it is necessary to check the boundary conditions of the stress field to ensure that they match the actual boundary conditions of the problem. When the crack temperatures far from the crack tips reach a uniform value of $T_0 - \theta_0 (\theta_0 > 0)$, then for distances far from the crack tips (large $|Y|$) the temperature and stress fields become uniform and the problem is essentially the same as uniformly lowering the temperature of the halfspace $Y < 0$ by an amount θ_0 while the temperature of the halfspace $Y > 0$ remains unchanged. Requiring that the displacement at $Y = \pm\infty$ be zero and also that there be no strain the x- and z-directions gives the stresses

$$\begin{Bmatrix} \sigma_{yy} \\ \sigma_{xx}(Y<0) \\ \sigma_{xx}(Y>0) \end{Bmatrix} = \frac{E\alpha\theta_0}{2(1-2\nu)} \begin{Bmatrix} 1 \\ \dfrac{(2-3\nu)}{(1-\nu)} \\ \dfrac{\nu}{(1-\nu)} \end{Bmatrix} \tag{12}$$

The stresses obtained for large $|Y|$ using the source superposition technique described in this paper are

$$\begin{Bmatrix} \sigma_{yy} \\ \sigma_{xx}(Y<0) \\ \sigma_{xx}(Y>0) \end{Bmatrix} = \frac{E_1}{2} \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix}; \frac{E\alpha\theta_0}{(1-\nu)} \equiv E_1. \tag{13}$$

These stresses (and the associated strains) are brought into agreement with those of eqn (12) when they are superimposed on the uniform stress state

$$\begin{Bmatrix} \sigma_{yy} \\ \sigma_{xx} \end{Bmatrix} = \frac{E\alpha\theta_0}{2(1-2\nu)} \begin{Bmatrix} \dfrac{\nu}{1-\nu} \\ 1 \end{Bmatrix}. \tag{14a, b}$$

Thus the total solution for the crack tractions is obtained by a superposition of the stresses determined from eqns (10) and those of eqn (14).

Examples of the heat source distribution and crack tractions obtained from the solution of eqn (10) are shown, for various values of normalized spacing, in Figs. 4 and 5 for the case in which the temperature of the crack surfaces in uniformly lowered by an amount θ_0. For small distances behind the crack tip, the heat fluxes are approximately those of a single constant-velocity, uniform temperature crack[8]:

$$-Q/\theta_0 = \sqrt{(2/\pi^3)}\,|y/b|^{-1/2}. \tag{15}$$

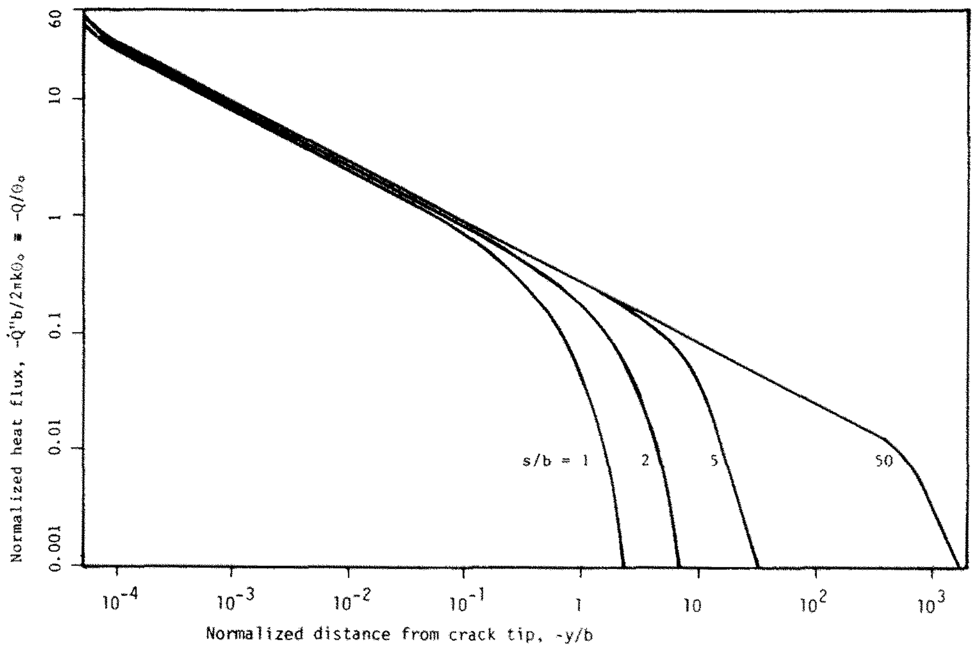

Fig. 4. Distribution of normalized heat flux along self-driven cracks for several values of normalized crack spacing.

For large spacings, the tractions in Fig. 5 for the near-tip region are compressive. However, superposition of the uniform stress state of eqn (14) gives final crack tractions which are tensile for all values of y, for all cases.

CONCLUSIONS

The procedure presented in this paper provides an efficient means of determining the heat fluxes and surface tractions for thermally self-driven cracks. This procedure has been used by the authors to analyze the possibility of self-driven cracking in Hot Dry Rock geothermal systems [9]. The crack tractions, calculated as a function of crack spacing and crack surface

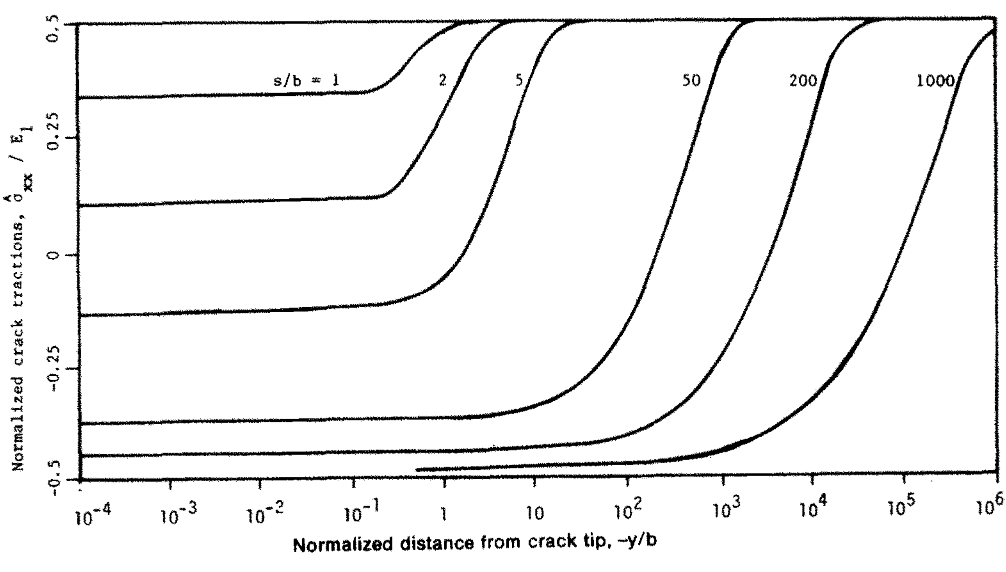

Fig. 5. Distribution of normalized crack traction along self-driven cracks for several values of normalized crack spacing.

temperature distribution, are used to compute the stress intensity factors which, in turn, determine steady-state crack velocities. The rate at which cooling fluid must flow through the crack is determined by the heat fluxes.

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